NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1012

VELOCITY DISTRIBUTION ON WING SECTIONS OF ARBITRARY SHAPE IN COMPRESSIBLE POTENTIAL FLOW

II - SUBSONIC SYMMETRIC ADIABATIC FLOWS

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SUMMARY

This paper extends the method of computing the pressure distribution along a symmertical profile of arbitrary shape given in NACA Technical Note No. 1006 under the assumption of a linearized pressure-volume relation to the case of an everywhere subsonic flow satisfying the rigorous adiabatic equation of state. Either the stream Mach number or the maximum local Mach number may be prescribed.

The actual applicability of the method depends upon the development of efficient procedures for numerical integration of linear partial differential equations with variable coefficients.

Tables and graphs of functions needed for the computation are given, and a velocity correction formula proposed by Garrick and Kaplan is discussed.

INTRODUCTION

In a recent paper (reference 1) the author presented an effective method of computing the velocity distribution along a symmetrical profile of arbitrary given shape under the assumption that the potential compressible flow past the profile obeys the so-called linearized pressure-density relation. In this paper this method is extended to the case of the adiabatic pressure-density relation which actual flows satisfy with a high degree of accuracy. In fact, the method presented here is applicable to any empirically or theoretically given pressure-density relation, provided the flow remains everywhere subsonic.

The computational labor involved in applying this method is considerably greater than in the case of the linearized pressure-density relation. The actual computations could hardly be attempted without using either automatic computing machines or electrical models. In fact, at the present state of mathematical analysis, it is doubtful whether reliable theoretical results on pressure distributions for high subsonic speeds and on the values of the critical Mach number for various profiles could be obtained by any method without extensive numerical computations. On the other hand, the present ideas concerning what constitutes difficult computations will doubtless undergo a very radical change in the near future due to the development of modern automatic computing machines.

In the author's opinion, the merit of the present method lies in the fact that it eliminates the mathematical difficulties peculiar to the fluid dynamical problem considered and reduces the solution of this problem to the numerical integration of a linear partial differential equation. This latter problem is of prime importance to all branches of applied mathematics and is the subject of many investigations.

The present method is based upon the application of a transformation which for compressible flows takes the place of conformal mapping. This transformation was introduced in a previous paper. (See reference 2.) The pressure distribution problem is first reduced to the determination of a function $f(\omega)$ defining a point-to-point correspondence between the prescribed profile and a circle. Then it is shown that this function satisfies a functional equation of the form $f = F(f)$ where F is a certain functional operator. In order to determine f, a trial solution f_0 is chosen, and the functions $f_1 = f(f_0)$, $f_2 = F(f_1), \ldots$ are computed. On the basis of computations carried out for a simple special case $(\gamma = -1)$, it may be expected that the sequence f_1, $f_2, \ldots$ rapidly converges toward the desired function $f(\omega)$. The computation of the transform $F(f_n)$ involves an integration of a linear partial differential equation previously mentioned.

On the basis of the theoretical results obtained, a discussion of a "velocity correction formula" proposed by Garrick and Kaplan is given.

In connection with this investigation certain computations were carried out by the Computation Project of the Bureau of Ships at Harvard University. The author expresses his sincere gratitude to the Bureau of Ships, to Commander H. H. Aiken, officer-in-charge of the project, and to Lieutenant H. A. Arnold and Mr. R. R. Seeber, Jr., who were in charge of the computations. The author also is indebted to Mr. Charles Saltzer for competent assistance.

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SYMBOLS

A, B, b	constants (appendix I)
$A(\zeta)$, $B(\zeta)$	auxiliary functions defined by equation (36)
a	speed of sound
C, C_1	positive constants
dS	non-Euclidean length element
E(P)	domain exterior to the profile P
$F[f(\omega'),\omega]$	functional transformation defined in section 8
$f(\omega)$	function defining the mapping of the circle into the profile
g_{ik}	coefficients of the metric generated by the flow
$H(\zeta)$	harmonic function defined by equations (42) and (43)
$h(\omega)$	function defined by equation (40)
M	Mach number
P	profile in the z-plane
P_i	lattice points in the Z-plane (appendix I)
p	pressure

Q	dimensionless speed q/a_o as function of q^{*2}
Q_1	dimensionless speed q/a_o as function of q^*
q	speed
q^*	distorted speed
r	modulus of ζ
S	length of the curve P
s	arc length measured along the profile
T	function defined by equation (4)
u, v	velocity components
u^*, v^*	components of the distorted velocity
w	complex velocity
w^*	distorted velocity
x, y	Cartesian coordinates in the z-plane
X, Y	Cartesian coordinates in the Z-plane
z	complex variable in the plane of the flow
z_L, z_T	leading and trailing edges
Z	auxiliary complex variable
α	angle at the trailing edge
β_i	coefficients of the difference equation (appendix I)
β_{ij}	coefficients of the Liebmann transformation
γ	exponent in the pressure-density relation
ζ	complex variable in the plane of the circle
Θ	slope of the profile P
θ	angle between the velocity vector and the x-axis

$\lambda(M_\infty)$	value of q^* for $M = M_\infty$
ξ, η	Cartesian coordinates in the ζ-plane
ρ	density
σ	dimensionless length parameter or the profile
τ	argument of ζ
χ	auxiliary function defined by equation (35)
φ	velocity potential
Φ	function proportional to the velocity potential
ψ	function proportional to the stream function
ω	argument of a point on the circle $\|\zeta\| = 1$
$(\ )_x$	partial derivative of $(\ )$ with respect to x (similarly for y, ξ, η, etc.)
$(\ )_o$	value of $(\ )$ at a stagnation point
$(\ )_\infty$	value of $(\ )$ at infinity
$(\ )_{max}$	maximum of $(\ )$ in the domain considered
$(\tilde{\ })$	boundary value of $(\ )$

ANALYSIS

1. Basic Relations

A steady two-dimensional potential flow of a compressible fluid is characterized by the quantities: p (pressure), ρ (density), q (local speed), and θ (angle between the velocity vector and some fixed direction, say the direction of the x-axis). The components of the velocity vector are given by

$$u = q \cos \theta, \quad v = q \sin \theta$$

the complex quantity

$$w = u - iv = qe^{-i\theta}$$

is called the (conjugate) complex velocity. Pressure and density are connected by the adiabatic relation

$$p/\rho^{\gamma} = p_o/\rho_o^{\gamma}$$

where the subscript o refers to the fluid at a stagnation point. The exponent γ is a constant; for air $\gamma = 1.405$. Instead of the speed q it is convenient to use the dimensionless quantities

$$\frac{q}{a_o} \quad \text{and} \quad M = \frac{q}{a}$$

where

$$a = \sqrt{\frac{dp}{d\rho}}$$

is the speed of sound.

Bernoulli's theorem implies the relations

$$\rho = \rho_o \left(1 - \frac{\gamma - 1}{2} \frac{q^2}{a_o^2}\right)^{\frac{1}{\gamma-1}} \tag{1}$$

$$p = p_o \left(1 - \frac{\gamma - 1}{2} \frac{q^2}{a_o^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$M^2 = \frac{q^2/a_o^2}{1 - \frac{\gamma - 1}{2} \frac{q^2}{a_o^2}} \tag{2}$$

It will be convenient to use the following two functions q^* and T of the speed which are defined in the subsonic range by

$$q^* = \exp \int_{q_s}^{q} \sqrt{1 - M^2}\,\frac{dq}{q} \tag{3}$$

and

$$T = \frac{\rho_o}{\rho}\sqrt{1 - M^2} \tag{4}$$

Here

$$q_s = a_o\sqrt{\frac{2}{\gamma + 1}}$$

is the critical speed for which $M = 1$. A simple computation yields

$$q^{*2} = \frac{\sqrt{1 - \frac{\gamma-1}{2}\frac{q^2}{a_o^2}} - \sqrt{1 - \frac{\gamma+1}{2}\frac{q^2}{a_o^2}}}{\sqrt{1 - \frac{\gamma-1}{2}\frac{q^2}{a_o^2}} + \sqrt{1 - \frac{\gamma+1}{2}\frac{q^2}{a_o^2}}}\left\{\frac{\gamma_1\sqrt{1 - \frac{\gamma-1}{2}\frac{q^2}{a_o^2}} + \sqrt{1 - \frac{\gamma+1}{2}\frac{q^2}{a_o^2}}}{\gamma_1\sqrt{1 - \frac{\gamma-1}{2}\frac{q^2}{a_o^2}} - \sqrt{1 - \frac{\gamma+1}{2}\frac{q^2}{a_o^2}}}\right\}^{\gamma_1} \tag{5}$$

$$T = \sqrt{\frac{1 - \frac{\gamma + 1}{2}\frac{q^2}{a_o^2}}{\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_o^2}\right)^{\gamma_1^2}}} \tag{6}$$

where

$$\gamma_1 = \sqrt{\frac{\gamma + 1}{\gamma - 1}}$$

It is clear that the "distorted speed" q^* is an increasing function of the dimensionless quantity q/a_o and that

$$q^* = 0 \quad \text{for} \quad q = 0, \qquad q^* = 1 \quad \text{for} \quad q = q_s$$

Hence all quantities characterizing the flow may be expressed as functions of q^{*2}; in particular

$$q = a_o Q(q^{*2}) \qquad T = T(q^{*2}) \tag{7}$$

The functions discussed in this section are tabulated (for $\lambda = 1.405$) in tables 1 to 4 and plotted in figures 1 to 3. Tables 1 and 2 on which the other computations were based were taken over from tables computed on the I. B. M. Automatic Sequence Controlled Calculator. (See reference 3.)

The complex function

$$w^* = q^* \, e^{-i\theta} \tag{8}$$

is called the distorted velocity. The knowledge of w^* implies that of w, since

$$\frac{w}{a_o} = Q(|w^*|^2) \quad \frac{w^*}{|w^*|}$$

<u>Remark 1</u>: For an isothermal flow ($\lambda = 1$) the preceding formulas must be replaced by the following

$$\rho = \rho_o e^{-q^2/2a_o^2}, \quad p = p_o e^{-q^2/2a_o^2}$$

$$M = q/a_o$$

$$q^* = \frac{q/a_o}{1 + \sqrt{1 - \frac{q^2}{a_o^2}}} \, e^{\sqrt{1 - q^2/a_o^2}}$$

$$T = \sqrt{1 - \frac{q^2}{a_o^2}} \, e^{q^2/2a_o^2}$$

<u>Remark 2</u>: The preceding formulas, with the exception of those for p, remain valid if the pressure-density relation is assumed in the slightly more general form

$$p = A\rho^{\gamma} + B$$

where A and B are constants.

<u>Remark 3</u>: The method described in this report can be applied to any pressure-density relation, provided the flow remains subsonic.

2. The Boundary Value Problem

Consider a symmetrical profile P in the plane of the complex variable $z = x + iy$. It will be assumed that the x-axis coincides with the axis of symmetry of the profile and that the trailing edge z_T is to the right. Let S be the length of the curve P and s the arc length measured along this curve from the point z_T in the counterclockwise direction. The points on the profile may be characterized by the dimensionless parameter

$$\sigma = 2\pi \frac{s}{S} \tag{9}$$

Thus $\sigma = 0$ corresponds to z_T and $\sigma = \pi$ to the leading edge z_L. Furthermore, let $\Theta(\sigma)$ denote the slope of P at a point corresponding to σ. This angle is defined as the positive angle by which the positive x-axis must be turned to move it parallel to the tangent vector to P pointing in the direction of increasing σ. (See fig. 5.) Thus the function $\Theta(\sigma)$ depends only upon the shape of P and

$$\Theta(0) = \pi - \alpha/2, \quad \Theta(\pi) = 3\pi/2, \quad \Theta(2\pi) = 2\pi + \alpha/2 \tag{10}$$

where α is the angle at the trailing edge $(0 \leq \alpha \leq \pi)$. Furthermore,

$$\Theta(2\pi - \sigma) = 3\pi - \Theta(\sigma) \tag{11}$$

The equation of the profile P can be written in the form

$$z = z_T + S\int_0^{\sigma} e^{i\Theta(\sigma')}d\sigma', \quad 0 \leq \sigma \leq 2\pi \tag{12}$$

Consider now a steady subsonic circulation-free potential flow of a compressible fluid past P. Let $a_0\varphi(x,y)$ be the velocity potential. Then

$$u = a_0 \frac{\partial\varphi}{\partial x} \quad \text{and} \quad v = a_0 \frac{\partial\varphi}{\partial y} \tag{13}$$

and the magnitude of the velocity vector q is given by

$$q = a_0 \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2} \tag{14}$$

so that the continuity equation

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_0}\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\rho_0}\frac{\partial\varphi}{\partial y}\right) = 0 \tag{15}$$

becomes a nonlinear partial differential equation for φ. In fact, this equation may be written in the form

$$\left(1 - \frac{\gamma+1}{2}\varphi_x{}^2 - \frac{\gamma-1}{2}\varphi_y{}^2\right)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy}$$

$$+ \left(1 - \frac{\gamma-1}{2}\varphi_x{}^2 - \frac{\gamma+1}{2}\varphi_y{}^2\right)\varphi_{yy} = 0 \tag{16}$$

where subscripts denote partial differentiation. For the special case $\gamma = -1$, this becomes the equation of a minimal surface.

The velocity potential $\varphi(x,y)$ is determined as the (one-valued) solution of this differential equation satisfying certain auxiliary conditions. The first of these states that the undisturbed flow has the positive x-direction. It may be written in the form

$$\lim_{z\to\infty} \frac{\partial\varphi}{\partial x} > 0, \qquad \lim_{z\to\infty} \frac{\partial\varphi}{\partial y} = 0 \tag{17}$$

Since

$$\tan \theta = \frac{v}{u} = \frac{\partial \varphi}{\partial y} \Big/ \frac{\partial \varphi}{\partial x} \tag{18}$$

this condition is equivalent to the following

$$\lim_{z \to \infty} \theta = 0 \tag{19}$$

The second condition is a boundary condition expressing the fact that the profile P is a streamline:

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad P \tag{20}$$

where $\partial/\partial n$ denotes differentiation in the normal direction. In view of the symmetry of the flow this condition may be restated as follows

$$\theta = \begin{Bmatrix} \Theta - \pi \\ \Theta + 2\pi \end{Bmatrix} \text{ on the } \begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix} \text{ bank of } P \tag{21}$$

Finally, a condition is needed which determines the magnitude of the speeds involved. This can be done in two ways. It is possible to prescribe the speed q_∞ of the undisturbed flow. The condition then reads

$$\lim_{z \to \infty} \frac{\partial \varphi}{\partial x} = \frac{q_\infty}{a_0} \tag{22}$$

or

$$\lim_{z \to \infty} q = q_\infty \tag{23}$$

Alternatively, it is possible to prescribe the maximum local speeds q_{max}. The condition reads

$$\max \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2} = \frac{q_{max}}{a_0} \tag{24}$$

or

$$\max q = q_{max} \tag{25}$$

In the first case, the stream Mach number M_∞ is prescribed, in the second, the maximum local Mach number M_{max}. In the second case, the condition imposed on φ has the character of a boundary condition, since the maximum speed is attained at the boundary. (The proof of this statement will be found in reference 4; it also follows from the considerations of sec. 4.)

In the following, this boundary value problem will be reduced to a mapping problem.

3. Mapping of the Profile into a Circle

The hodograph of the compressible flow around P is the domain in the w-plane $(w = u - iv)$ into which the domain E(P) exterior to P is taken by the transformation

$$w = a_0\left(\frac{\partial\varphi}{\partial x} - i\,\frac{\partial\varphi}{\partial y}\right) \tag{26}$$

In the case of a symmetrical flow the hodograph is a doubly covered Riemann surface with a branch point at $w = q_\infty$ as shown in figure 5. This surface is transformed into the "distorted hodograph" by the transformation

$$w^* = q^*(|w|/a_0)\,\frac{w}{|w|} \tag{27}$$

where q^* is the function of $q = |w|$ defined by (3). Finally, the distorted hodograph is mapped conformally into the domain $|\zeta| \geq 1$ of the auxiliary ζ-plane by an analytic function $\zeta(w^*)$ satisfying the conditions

$$\zeta = \infty, \quad \frac{d\zeta}{dw^*} > 0 \quad \text{for} \quad w^* = q^*_\infty = q^*(q_\infty/a_0)$$

In this way a transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \tag{28}$$

of the domain $E(P)$ into the domain $|\zeta| > 1$ is obtained. It is seen that the points

$$z = \infty, \quad z = z_L, \quad z = z_T$$

are taken into the points

$$\zeta = \infty, \quad \zeta = -1, \quad \zeta = 1$$

respectively, and that the horizontal direction at infinity is preserved by the mapping:

$$\lim_{z \to \infty} \frac{\partial \xi}{\partial x} > 0, \quad \lim_{z \to \infty} \frac{\partial \xi}{\partial y} = 0, \quad \lim_{z \to \infty} \frac{\partial \eta}{\partial x} = 0, \quad \lim_{z \to \infty} \frac{\partial \eta}{\partial y} > 0 \tag{29}$$

The resulting mapping of the profile P into the unit circle can be described by means of a function

$$\sigma = f(\omega) \tag{30}$$

such that a point z of P corresponding to the parameter value $\sigma = f(\omega)$ is taken into the point $\zeta = e^{i\omega}$. The function (30) is an increasing function and

$$f(0) = 0, \quad f(2\pi) = 2\pi \tag{31}$$

Furthermore, by reasons of symmetry

$$f(2\pi - \sigma) = 2\pi - f(\sigma) \tag{32}$$

so that

$$f(\pi) = \pi \tag{33}$$

It will be shown that the knowledge of the function $f(\omega)$ implies that of the velocity and pressure distribution along P.

<u>Remark 1</u>: It has been shown (reference 2) that the transformation (28) is conformal with respect to the following Riemann metric:

$$dS^2 = g_{11}\, dx^2 + 2g_{12}dxdy + g_{22}dy^2 \qquad (34)$$

where

$$\left.\begin{aligned} g_{11} &= 1 - M^2 \sin^2 \theta \\ g_{12} &= M^2 \sin \theta \cos \theta \\ g_{22} &= 1 - M^2 \cos^2 \theta \end{aligned}\right\} \qquad (35)$$

It is instructive to consider three special cases. For an infinitely slow flow (i.e., for an incompressible fluid) $M = 0$ and $dS^2 = dx^2 + dy^2$. Then the mapping just constructed is the standard conformal mapping of a profile into a circle. For a uniform flow in the x-direction $\theta = 0$, M is constant and $dS^2 = dx^2 + (1 - M^2)dy^2$. The introduction of the metric (34) is equivalent to the well known Prandtl-Glauert contraction. Finally, in the case $\gamma = -1$, the mapping (28) coincides with the mapping defined by the equation

$$\text{constant}\left(\zeta + \frac{1}{\zeta}\right) = \varphi + i\psi$$

where ψ is the stream function of the flow. (See reference 1.)

4. Computation of the Velocity in the ζ-Plane

From the way in which the mapping of the z-plane into the ζ-plane has been defined it follows that the distorted velocity

$$w^* = q^* e^{-i\theta}$$

is a one-valued regular analytic function of ζ for $|\zeta| > 1$. It vanishes only for $\zeta = -1$ and (unless $\alpha = 0$) for $\zeta = 1$. Moreover, the function

$$\chi(\zeta) = \frac{w^* \zeta^{1+\alpha/\pi}}{(\zeta + 1)(\zeta - 1)^{\alpha/\pi}}$$

is continuous for $|\zeta| \leq 1$ and is nowhere in this domain equal to either zero or infinity. (For the proof of this last assertion, see reference 1, sec. 3.)

Set

$$\log \chi = A + iB \tag{36}$$

Then $A + iB$ and hence $B - iA$ are regular one-valued analytic functions. Hence $A(\zeta)$ may be computed in terms of the boundary values of the function $B(\zeta)$. For $|\zeta| > 1$ this is done by means of the Poisson-Schwarz integral formula

$$B(\zeta) - iA(\zeta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\omega} + \zeta}{e^{i\omega} - \zeta} B(e^{i\omega})d\omega - iA(\infty)$$

which implies that

$$A(\zeta) = \frac{1}{\pi}\int_0^{2\pi} \frac{r \sin(\tau - \omega)}{1 - 2r \cos(\tau - \omega) + r^2} B(e^{i\omega}) + A(\infty), \quad (\zeta = re^{i\tau})$$

For $|\zeta| = 1$ the formula to be used reads

$$A(e^{i\omega}) = -\frac{1}{2\pi}\int_0^{\pi} \left\{B(e^{i\omega + i\tau}) - B(e^{i\omega - i\tau})\right\} \cot \frac{\tau}{2}\, d\tau + A(\infty)$$

(See, for instance, reference 5, p. 243.) Of course, it is also possible to determine first $A(e^{i\omega})$ and then to compute $A(re^{i\tau})$, $r > 1$ by the formula

$$A(re^{i\tau}) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\tau - \omega) + r^2} A(e^{i\omega})d\omega$$

If the value of θ at a point of $\tilde{P}$ corresponding to the parameter value σ is denoted by $\tilde{\theta}(\sigma)$,

$$B(e^{i\omega}) = -\tilde{\theta}[f(\omega)] - \arg(1+e^{i\omega}) - \frac{\alpha}{\pi}\arg(e^{i\omega}-1) + \left(1+\frac{\alpha}{\pi}\right)\omega \quad (37)$$

and

$$\arg(1+e^{i\omega}) = \begin{cases} \omega/2 & \text{for} \quad 0 < \omega < \pi \\ \omega/2+\pi & \text{for} \quad \pi < \omega < 2\pi \end{cases}$$

$$\arg(e^{i\omega}-1) = \frac{\omega}{2} + \frac{\pi}{2}$$

Furthermore, by virtue of (36) and (21)

$$\tilde{\theta}[f(\omega)] = \left\{\begin{matrix} \Theta[f(\omega)] - \pi & \text{for} \quad 0 < \omega < \pi \\ \Theta[f(\omega)] - 2\pi & \text{for} \quad \pi < \omega < 2\pi \end{matrix}\right\} \quad (38)$$

Also

$$\log q^* = A + \log\left|\frac{(\zeta+1)(\zeta-1)^{\frac{\alpha}{\pi}}}{\zeta^{1+\alpha/\pi}}\right|$$

and if the value of q^* at a point of P belonging to the parameter value σ is denoted by $\tilde{q}^*(\sigma)$, then

$$\log \tilde{q}^*[f(\omega)] = A(e^{i\omega}) + \log\left\{2^{1+\frac{\alpha}{\pi}}\left|\cos\frac{\omega}{2}\right|\left|\sin\frac{\omega}{2}\right|^{\frac{\alpha}{\pi}}\right\}$$

By noting that $A(\infty)$ equals the logarithm of the value of the distorted speed at infinity and combining the preceding formulas, the following expressions for q^* are obtained:

$$\tilde{q}^*\left[f(\omega)\right] = q_\infty^*\, 2^{1+\frac{\alpha}{\pi}}\left|\cos\frac{\omega}{2}\right|\left|\sin\frac{\omega}{2}\right|^{\frac{\alpha}{\pi}} e^{h(\omega)} \quad (39)$$

where

$$h(\omega) = \frac{1}{2\pi}\int_0^{\pi}\left\{\Theta[f(\omega+\tau)] - \Theta[f(\omega-\tau)]\right\}\cot\frac{\tau}{2}\,d\tau \quad (40)$$

and

$$q^*(\zeta) = q^*_\infty \left| \frac{(\zeta+1)(\zeta-1)^{\frac{\alpha}{\pi}}}{\zeta^{1+\alpha/\pi}} \right| e^{H(\zeta)} \tag{41}$$

where the (harmonic) function $H(\zeta)$ is given by either of the two equivalent formulas:

$$H(\zeta) = \frac{1}{\pi}\int_0^{2\pi} \frac{r \sin(\tau-\omega)}{1-2r\cos(\tau-\omega)+r^2} \left\{ \Theta[f(\omega)] - \frac{\alpha+\pi}{2\pi}\omega \right\} d\omega \tag{42}$$

or

$$H(\zeta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\tau-\omega)+r^2} h(\omega)d\omega \tag{43}$$

In both cases $\zeta = re^{i\tau}$.

If in the original boundary value problem the value of the speed at infinity has been prescribed, then q^*_∞ is a known constant. If, however, the value of the maximum local speed is prescribed, then q^*_∞ is determined from the relation

$$2^{1+\frac{\alpha}{\pi}} q^*_\infty = q^*_{max} \Big/ \max_{0\leq\omega\leq 2} \left\{ \left|\cos\frac{\omega}{2}\right| \left|\sin\frac{\omega}{2}\right|^{\frac{\alpha}{\pi}} e^{h(\omega)} \right\} \tag{44}$$

where q^*_{max} denotes the (given) maximum local distorted speed.

The function θ also is uniquely determined by $f(\omega)$, for being a harmonic function of ξ and η and satisfying conditions (38) it is given by the formula

$$\theta(\zeta) = -\frac{1}{2\pi}\int_0^{\pi} \frac{1-r^2}{1-2r\cos(\tau-\omega)+r^2} \left\{\Theta[f(\omega)] - \pi\right\} d\omega$$

$$-\frac{1}{2\pi}\int_\pi^{2\pi} \frac{1-r^2}{1-2r\cos(\tau-\omega)+r^2} \left\{\Theta[f(\omega)] - 2\pi\right\} d\omega \tag{45}$$

It should be observed that

$$\lim_{\zeta \to \infty} \theta = 0 \tag{46}$$

No difficulties are involved in computing the abovementioned functions since the integral defining the function $h(\omega)$ is a proper Riemann integral. In fact, for $\tau = 0$ the integrand is equal to

$$4\Theta' [f(\omega)] f'(\omega) - \frac{2(\alpha + \pi)}{\pi}$$

In computing this integral the expression in the braces must be treated as a periodic function of ω with the period 2π.

After q^* and θ have been determined the quantities q/a_0, T, ρ, p, and g_{ik} can be computed as functions of ξ and η. In particular, the boundary values $\tilde{q}(\sigma)$ of q are given by

$$\tilde{q}\,[f(\omega)] = a_0\, Q\left\{\tilde{q}^*\,[f(\omega)]^2\right\}$$

Thus the statement that the knowledge of the function $\sigma = f(\omega)$ yields the pressure distribution along P is verified.

In the preceding considerations no properties of the function $f(\omega)$ except the boundary conditions (31) were used. Therefore the following statement is valid.

To every function $f(\omega)$ (which is not necessarily the mapping function) satisfying the conditions $f(0) = 0$ and $f(2\pi) = 2\pi$ there corresponds an analytic function $w^* = q^* e^{-i\theta}$ defined by equations (39) to (45). This function is uniquely determined by $f(\omega)$, the function $\Theta(\sigma)$ which depends only upon the shape of the profile P and either of the two parameters q^*_∞ and q^*_{max} which are connected by relation (44).

If $|w^*| < 1$ everywhere, then the functions q/a_0 and T also can be computed as functions of ζ. This will always be the case if the prescribed value of $q^*_{max} < 1$.

If the function $f(\omega)$ also satisfies condition (31) (as the actual mapping function does), then the analytic function w^* will satisfy the condition

$$w^*(\bar{\zeta}) = \overline{w^*(\zeta)} \tag{47}$$

(The bar denotes the conjugate complex quantity.) In this case the function $T = T(\xi,\eta)$ corresponding to $f(\omega)$ will possess a symmetry with respect to the ξ-axis:

$$T(\xi,-\eta) = T(\xi,\eta) \tag{48}$$

In the following condition (31) will always be assumed.

5. Computation of the Potential in the ζ-Plane

It is known that the potential φ considered as a function of u^* and v^* $(w^* = u^* - iv^*)$ satisfies the linear partial differential equation

$$\frac{\partial}{\partial u^*}\left(\frac{1}{T}\frac{\partial\varphi}{\partial u^*}\right) + \frac{\partial}{\partial v^*}\left(\frac{1}{T}\frac{\partial\varphi}{\partial v^*}\right) = 0 \tag{49}$$

(See reference 2, p. 15.) This elliptic equation is invariant with respect to conformal transformation so that φ considered as a function defined in the ζ-plane satisfies the equation

$$\frac{\partial}{\partial \xi}\left(\frac{1}{T}\frac{\partial\varphi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{T}\frac{\partial\varphi}{\partial \eta}\right) = 0 \tag{50}$$

To determine the boundary conditions satisfied by φ observe that by virtue of the conformality of the mapping (28) with respect to the metric (34) a line element normal to the profile P is taken into a line element normal to the circle $|\zeta| = 1$. Hence condition (20) implies that in the ζ-plane

$$\frac{\partial}{\partial r}\varphi(r\cos\tau,\ r\sin\tau)\Big|_{r=1} = 0 \tag{51}$$

Next

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial x}, \qquad \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial y}$$

From (17) and (29) it follows that

$$\left.\begin{aligned} &\lim_{\zeta\to\infty}\frac{\partial\varphi}{\partial\xi}=\frac{q_\infty}{a_0}\bigg/\left(\lim_{\zeta\to\infty}\frac{\partial\xi}{\partial x}\right)>0\\ &\lim_{\zeta\to\infty}\frac{\partial\varphi}{\partial\eta}=0\end{aligned}\right\}\tag{52}$$

From the symmetry of the flow (in fact, from condition (48)) it also follows that

$$\varphi(\xi,-\eta)=\varphi(\xi,\eta)\tag{53}$$

Now let $\Phi(\xi,\eta)$ denote the (one-valued) solution of equation (50) satisfying the conditions

$$\frac{\partial}{\partial r}\Phi\bigg|_{r=1}=0\tag{54}$$

and

$$\lim_{\zeta\to\infty}\frac{\partial\Phi}{\partial\xi}=1,\qquad\lim_{\zeta\to\infty}\frac{\partial\Phi}{\partial\eta}=0\tag{55}$$

Then there exists a positive constant C such that

$$\varphi=C\,\Phi\tag{56}$$

This follows from the linearity and homogeneity of the differential equations and the boundary conditions.

The existence of the function Φ is assured by known theorems and in principle Φ can be computed numerically with any desired degree of accuracy. The actual computation of the function Φ however, presents considerable difficulties. A brief discussion of the various possibilities will be found in appendix I. For the method described in this report it is irrelevant which numerical or mechanical device is actually used for solving equation (50).

Thus it is seen that the knowledge of the function $f(\omega)$ permits the computation of the potential φ as a function of ξ, except for a constant positive factor.

Furthermore, to every function $\sigma = f(\omega)$ satisfying conditions (31) and (32) there corresponds a function $\Phi(\xi,\eta)$ satisfying the condition

$$\Phi(\xi,-\eta) = \Phi(\xi,\eta) \tag{57}$$

namely, the solution of (50) under the conditions (54) and (55) where $T = T(|w^*|^2)$ and $w^* = q^* e^{-i\theta}$ is the analytic function associated with $f(\omega)$ by means of equations (39) to (45). The function Φ can be computed if $|w^*|$ nowhere exceeds unity and is determined by the shape function $\Theta(\sigma)$ and by either of the two parameters q^*_∞ or q^*_{max}. The actual determination of Φ requires the numerical integration of a linear partial differential equation in a domain independent of the particular problem.

6. The Functional Equation Satisfied by $f(\omega)$

Along the profile P the potential φ and the speed q are connected by the relation

$$q = a_0 \left|\frac{d\varphi}{ds}\right|$$

or

$$|ds| = \frac{a_0}{q}\,|d\varphi|$$

If the function Φ defined in the previous section is introduced, this can be written in the form

$$ds = \frac{a_0 C}{\tilde{q}\,[f(\omega)]}\left|\frac{d}{d\omega}\Phi\,(\cos\omega,\ \sin\omega)\right| d\omega \tag{58}$$

Setting

$$\Phi(\cos\omega,\ \sin\omega) = \tilde{\Phi}(\omega) \tag{59}$$

integration yields

$$s = a_o\, C \int_o^{\omega} \frac{\left|\tilde{\Phi}'(\omega')\right| d\omega'}{\tilde{q}[f(\omega')]}$$

But $s = \sigma S/2\pi$, $\tilde{q} = Q(\tilde{q}^{*2})$, and $\sigma = f(\omega)$.

Therefore

$$f(\omega) = \frac{SC}{2\pi} \int^{\omega} \frac{d\Phi(\omega')}{\tilde{Q}(\omega')}$$

where

$$\tilde{Q}(\omega) = Q\left\{\tilde{q}^*\,[f(\omega)]^2\right\}$$

By setting $\omega = 2\pi$ it is seen that

$$\frac{SC}{2\pi} = 2\pi \Big/ \int_o^{2\pi} \frac{d\tilde{\Phi}(\omega)}{\tilde{Q}(\omega)}$$

so that finally

$$f(\omega) = 2\pi \frac{\displaystyle\int_o^{\omega} \frac{d\tilde{\Phi}(\omega')}{\tilde{Q}(\omega')}}{\displaystyle\int_o^{2\pi} \frac{d\tilde{\Phi}(\omega')}{\tilde{Q}(\omega')}} \tag{60}$$

Since starting from the function $f(\omega)$ it is possible to compute $q^*(\zeta)$ and therefore also $\tilde{q}^*\,[f(\omega)]$, $\tilde{Q}(\omega)$, $T(\xi,\eta)$ and Φ, equation (60) is a functional equation satisfied by the function $f(\omega)$

7. Equivalence of the Functional Equation and the Boundary Value Problem

In this section it will be shown that a solution of the functional equation actually yields a solution of the boundary value problem formulated in section 2. In the same time it will be shown how the knowledge of the function $f(\omega)$ permits the computation of the velocity distribution at points not situated on the profile P.

In the following it is assumed that a function $f(\omega)$ satisfying (60) is known. This already implies that f is an increasing function (since its derivative is necessarily non-negative) and that the analytic function $w^* = q^* e^{-i\theta}$ associated with $f(\omega)$ (cf. sec. 4) satisfies the condition

$$|w^*| < 1 \tag{61}$$

The function $f(\omega)$ determines functions q/a_o, M, ρ/ρ_o, T, θ as functions of ξ and η. Set

$$\left.\begin{aligned} x &= C\int\frac{a_o}{q}\left\{\left(\cos\theta\,\Phi_\xi + \frac{\sin\theta}{\sqrt{1-M^2}}\Phi_\eta\right)d\xi \right.\\ &\qquad \left. + \left(\cos\theta\,\Phi_\eta - \frac{\sin\theta}{\sqrt{1-M^2}}\Phi_\xi\right)d\eta\right\} \\ y &= C\int\frac{a_o}{q}\left\{\left(\sin\theta\,\Phi_\xi - \frac{\cos\theta}{\sqrt{1-M^2}}\Phi_\eta\right)d\xi \right.\\ &\qquad \left. + \left(\sin\theta\,\Phi_\eta + \frac{\cos\theta}{\sqrt{1-M^2}}\Phi_\xi\right)d\eta\right\} \end{aligned}\right\} \tag{62}$$

where Φ is the function defined in section 5, subscripts denote partial differentiation, and C is a positive constant. These line integrals are independent of the path. (For the proof, see appendix II, sec. A). Hence equation (62) defines a transformation

$$x = x(\xi,\eta), \qquad y = y(\xi,\eta) \tag{63}$$

This transformation can also be written in the form

$$z = x + iy$$

$$= C\int \frac{a_o}{q}\, e^{i\theta}\, d\Phi - iC\int \frac{a_o}{q\sqrt{1-M^2}}\, e^{i\theta}\,(\Phi_\eta\, d\xi - \Phi_\xi\, d\eta) \tag{64}$$

In order to compute the image of the circle $|\zeta| = 1$ under this transformation the integration may be performed along this circle. But on the circle the normal derivative of Φ vanishes so that $\Phi_\eta\, d\xi - \Phi_\xi\, d\eta = 0$ and

$$\theta = \begin{cases} \Theta[f(\omega)] - \pi & \text{for} \quad 0 < \omega < \pi \\ \Theta[f(\omega)] - 2\pi & \text{for} \quad \pi < \omega < \pi \end{cases}$$

$$\frac{a_o}{q} = \tilde{Q}(\omega)$$

$$d\Phi = \begin{cases} - \left|\tilde{\Phi}'\right|\, d\omega & \text{for} \quad 0 < \omega < \pi \\ + \left|\tilde{\Phi}'\right|\, d\omega & \text{for} \quad \pi < \omega < 2\pi \end{cases}$$

so that the image of $|\zeta| = 1$ is the curve

$$z = C\int_0^\omega e^{i\Theta\,[f(\omega')]}\, \frac{|d\Phi\;(\omega')|}{\tilde{Q}(\omega')}, \qquad 0 \leq \omega \leq 2\pi$$

or, by (60)

$$z = C_1\int_0^\omega e^{i\Theta\,[f(\omega')]}\; f'(\omega')\, d\omega, \qquad 0 \leq \omega \leq 2\pi$$

where C_1 is a new positive constant. Thus, setting $\sigma = f(\omega)$

$$z = C_1 \int_0^{\sigma} e^{i\Theta(\sigma')} \, d\sigma', \qquad 0 \leq \sigma \leq 2 \tag{65}$$

Comparison of this with (12) shows that (62) takes the circle $|\zeta| = 1$ into the profile P, and, since $f(\omega)$ is an increasing function, the mapping is one-to-one. The constant C_1 can be determined from the length of P, since this length is equal to

$$S = C_1 \; 2\pi \tag{66}$$

Next, the Jacobian of the transformation (62) equals

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{a^2{}_0}{q^2} \frac{1}{\sqrt{1 - M^2}} \left\{ \Phi_\xi{}^2 + \Phi_\eta{}^2 \right\} \tag{67}$$

It can be shown that the gradient of Φ does not vanish for $|\zeta| > 1$. Hence the Jacobian is positive for $|\zeta| > 1$. Finally, by virtue of (55) and (46) as $\zeta \rightarrow \infty$,

$$\frac{\partial x}{\partial \xi} \longrightarrow C \frac{a_0}{q_\infty}, \; \frac{\partial x}{\partial \eta} \longrightarrow 0, \; \frac{\partial y}{\partial \xi} \longrightarrow 0, \; \frac{\partial y}{\partial \eta} \longrightarrow C \frac{a_0}{q_\infty} \frac{1}{\sqrt{1 - M_\infty{}^2}} \tag{68}$$

where M_∞ is the value of M at $\zeta = \infty$. Hence

$$\lim_{\zeta \rightarrow \infty} z = \infty \tag{69}$$

From the foregoing statements it follows that the transformation (62) yields a one-to-one mapping of the domain $|\zeta| > 1$ into the domain E(P). Hence all functions defined in the ζ-plane q/a_0, θ, Φ, and so forth, can be considered as functions of x and y. The transformation inverse to (63)

$$\xi = \xi(x,y), \qquad \eta = \eta(x,y)$$

is conformal with respect to the metric (34). The proof of this assertion will be found in appendix II, section B. Since Φ satisfies equation (50) it follows from a lemma proved in a previous paper (reference 2; appendix, converse to Lemma 1) that in the z-plane Φ satisfies equation (15) considered as a linear differential equation (since ρ/ρ_o is a known function in the ζ-plane and therefore also in the z-plane).

However, the relations

$$\frac{\partial\Phi}{\partial\xi} = \frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial\xi}$$

$$\frac{\partial\Phi}{\partial\eta} = \frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial\eta}$$

together with (62) yield the equations

$$\Phi_\xi = C\,\frac{a_o}{q}\left\{\left[\cos\,\theta\Phi_\xi + \frac{\sin\,\theta}{\sqrt{1 - M^2}}\,\Phi_\eta\right]\Phi_x + \left[\sin\,\theta\Phi_\xi - \frac{\cos\,\theta}{\sqrt{1 - M^2}}\,\Phi_\eta\right]\Phi_y\right\}$$

$$\Phi_\eta = C\,\frac{a_o}{q}\left\{\left[\cos\,\theta\Phi_\eta - \frac{\sin\,\theta}{\sqrt{1 - M^2}}\,\Phi_\xi\right]\Phi_x + \left[\sin\,\theta\Phi_\eta + \frac{\cos\,\theta}{\sqrt{1 - M^2}}\,\Phi_\xi\right]\Phi_y\right\}$$

Solving for Φ_x and Φ_y yields

$$\Phi_x = \frac{1}{C}\,\frac{q}{a_o}\cos\,\theta, \qquad \Phi_y = \frac{1}{C}\,\frac{q}{a_o}\sin\,\theta$$

Thus

$$u = q\cos\,\theta, \qquad v = q\sin\,\theta$$

are the components of the gradient of the function

$$a_o\varphi(x,y) = C\,a_o\,\Phi$$

The function φ satisfies equation (15) considered as a nonlinear equation, that is, it satisfies equation (16). Thus φ is the potential of a compressible flow in the z-plane and $u - iv = q\, e^{-i\theta}$ its complex velocity.

It is now easy to verify that φ also satisfies the boundary conditions. In fact, along P the angle θ coincides with the slope of the profile and at infinity θ vanishes. Finally, q takes the value $q_\infty = a_0\, Q(q^{*}{}_\infty{}^2)$ at infinity and its maximum is $q_{max} = a_0\, Q(q^{*2}{}_{max})$.

The foregoing discussion contains both the proof of the equivalence between the boundary value problem and the functional equation (60), and the description of a method permitting the computation of the flow in the whole domain E(P) once the function $f(\omega)$ is known.

8. Solution of the Functional Equation

In the preceding sections the determination of the pressure distribution along a given symmetrical profile P has been reduced to the determination of a function $\sigma = f(\omega)$ satisfying the conditions

$$\left.\begin{aligned} &f(0) = 0, \qquad\qquad f(2\pi) = 2\pi \\ &f(2\pi - \sigma) = 2\pi - f(\sigma) \end{aligned}\right\} \tag{70}$$

and the functional equation (60). The form of this equation suggests the application of the iteration method.

Let $f(\omega)$ be any function satisfying conditions (70). By use of this function the functions w^*, $\mathfrak{Q}(\omega)$, T and $\mathfrak{F}(\omega)$ can be computed (cf. secs. 4 and 5) provided the function w^* satisfies the condition $|w^*| < 1$. (This will always be the case if q^*_{max} is prescribed.) Therefore the right-hand side of (60) can be computed and will represent an increasing function $g(\omega)$ which will again satisfy conditions (70). This function will be denoted by

$$g(\omega) = F\, [f(\omega'),\, \omega] \tag{71}$$

With this notation equation (60) reads

$$f(\omega) = F\, [f(\omega'),\, \omega] \tag{72}$$

Thus the problem consists of finding a fixed point of the functional transformation F.

Now let $f_o(\omega)$ (approximation of order zero) be a function satisfying conditions (70) and set

$$\left.\begin{aligned} f_1(\omega) &= F\,[f_o(\omega'),\ \omega] \\ f_2(\omega) &= F\,[f_1(\omega'),\ \omega] \\ &\dots\dots\dots\dots \\ f_n(\omega) &= F\,[f_{n-1}(\omega'),\ \omega] \\ &\dots\dots\dots\dots \end{aligned}\right\} \tag{73}$$

If the sequence $f_o(\omega)$, $f_1(\omega)$, . . ., $f_n(\omega)$, . . . converges to a function $f(\omega)$ and if $\lim F(_n) = F(f)$, then $f(\omega)$ is a solution of equation (60).

It seems to be rather difficult to prove rigorously the convergence of this method. However, on the basis of computations carried out for the special case $\gamma = -1$ a fairly rapid convergence of the iterations seems probable, and it might be expected that good results will be obtained after a few steps. The real difficulty in computing the successive approximations lies in the numerical integration of the differential equation satisfied by Φ. (Cf. appendix I.) Thus the practical applicability of this method hinges upon the development of efficient methods for solving this problem, which, of course, is of prime importance for practically all branches of engineering mathematics.

It should be noted that the iteration process may break down if in the original boundary value problem the value of q_∞ is prescribed. This cannot happen if the value of q_{max} is prescribed.

The rapidity of the convergence will depend upon the appropriate choice of the function $f_o(\omega)$ (the approximation of order zero). Several possibilities of choosing f_o are listed in order of preference.

(1) Choose for f_o the solution of equation (60) previously obtained for a profile P' close to the desired profile and for a value of q'_∞ (or q'_{max}) close to the desired value of q_∞ (or q_{max}).

(2) Choose for f_o the solution of equation (60) for the special case $\gamma = -1$. (Cf. next sec.)

(3) Choose for f_o the function resulting from the conformal mapping of the profile into a circle.

(4) For thin profiles $f_o(\omega)$ may be chosen as

$$\left.\begin{aligned} f_o(\omega) &= \frac{\pi}{2}(1 - \cos\omega), \qquad 0 \leq \omega \leq \pi \\ f_o(2\pi - \omega) &= 2\pi - f(\omega) \end{aligned}\right\} \tag{74}$$

Note that (3) and (4) are special cases of (1). (In the case (3) $P' = P$ and $q'_\infty = 0$, in the case (4) P' is a straight segment and $q'_\infty = 0$.)

9. Comparison with the Case $\gamma = -1$

If it is assumed that the fluid obeys the so-called linearized pressure-density relation of Chaplygin-Kármán-Tsien (references 4, 6, and 7), $\gamma = -1$ and equations (1), (2), and (4) yield

$$T \equiv 1$$

$$\frac{a_o}{q} = \frac{1}{2}\left(\frac{1}{q^*} - q^*\right)$$

It is seen that in this case the function Φ becomes independent of the particular function $f(\omega)$. In fact,

$$\Phi = \xi + \frac{\xi}{\xi^2 + \eta^2}$$

so that

$$\tilde{\Phi}(\omega) = 2 \cos \omega$$

Equation (60) takes the form of an integral equation treated in a previous paper (reference 1, equation (51)). Since Φ does not have to be determined, only functions of one variable enter in the problem and the computational labor is greatly diminished. The convergence of the iteration method described in the preceding section when applied to this integral equation turned out to be very satisfactory.

10. Comparison with Other Methods

All other analytical methods proposed for solving the boundary value problem of section 2 involve an expansion of the potential function $\varphi(x,y)$ in terms of a parameter characterizing either the deviation of the flow from the uniform flow (such as the thickness parameter of a profile) or the deviation of the flow from an incompressible flow (such as the stream Mach number). These methods also involve successive integrations of linear partial differential equations. Methods based on such expansions necessarily suffer from the disadvantage that the convergence becomes worse as the deviation from the "undisturbed state" increases, that is, precisely as the influence of compressibility becomes more pronounced. In the present method the incompressible (i.e., infinitely slow) flow or the uniform flow (flow past a straight segment) is in no way distinguished, and the rate of convergence should not depend too much upon the value of the parameters q_∞ or q_{max}.

Furthermore, in all other methods the speed at infinity and therefore the stream Mach number are prescribed, and there is no way of telling whether this value of the stream Mach number does not exceed the critical Mach number. If this is the case, the flow becomes partly supersonic, and the convergence usually breaks down. In the present method it is possible to prescribe the maximum local Mach number M_{max} and the value of M_{max} may be chosen arbitrarily close to unity.

The preceding remark applies also to a purely numerical treatment of the nonlinear equation (15) (see a recent paper by Emmons, reference 8, and the literature quoted therein). This treatment consists in replacing the differential equation by a nonlinear difference equation which is solved by the relaxation method.

11. On a Velocity Correction Formula by Garrick and Kaplan

The functional equation to which the pressure distribution problem has been reduced throws new light on a velocity correction formula recently proposed by Garrick and Kaplan. (See reference 9.) These authors derived their "geometric mean formula" using an analogy between compressible and incompressible flows and some ideas from the theory of sigma-monogenic functions. In order to write this correction formula in the notations of the present paper, let $\lambda(M_\infty)$ denote the value of q^*_∞ corresponding to the stream Mach number M_∞ and set

$$Q_1(q^*) = Q(q^{*2}) \tag{75}$$

Then the Garrick-Kaplan formula reads

$$\left(\frac{q}{q_\infty}\right)_c = \frac{Q_1\left[\lambda(M_\infty)\left(\frac{q}{q_\infty}\right)_i\right]}{Q_1\left[\lambda\ (M_\infty)\right]} \tag{76}$$

Here $(q/q_\infty)_c$ denotes the ratio of the speed at a point of a profile to the speed at infinity for a compressible flow of stream Mach number M_∞ and $(q/q_\infty)_i$ is the same ratio for an incompressible flow (i.e., for a flow with $M_\infty = 0$). For the case of a fluid obeying the linearized pressure-density relation with $\gamma = -1$ formula (76) takes the form

$$\left(\frac{q}{q_\infty}\right)_c = \left(\frac{q}{q_\infty}\right)_i \frac{1 - \lambda^2}{1 - \lambda^2 (q/q_\infty)_i} \tag{77}$$

where

$$\lambda = \lambda(M_\infty) = \frac{M_\infty}{1 + \sqrt{1 - M_\infty{}^2}} \tag{78}$$

This is the Kármán-Tsien velocity correction formula. (See reference 6.)

It is immediately seen that formula (76) would be rigorously correct if the function $\sigma = f_o(\omega)$ resulting from the conformal mapping of the profile into a circle were the solution of the functional equation (60) for the desired value of M_∞. The situation may be summed up as follows: The difference between the relative velocity distribution (the distribution of the values q/q_∞) in an incompressible flow and in a compressible flow of stream Mach number M_∞ is due (a) to the difference between the values of the corresponding mapping functions $\sigma = f(\omega)$, and (b) to the fact that the way in which the velocity distribution is determined by the mapping function $f(\omega)$ depends upon the stream Mach number. The velocity correction formula (76) takes into account the factor (b) but fails to include the effect of factor (a).

On the basis of computations carried out for the case $\gamma = -1$ (see reference 1, figs. 1 and 2) it may be concluded that formula (76) will yield values of q/q_∞ which are too low. Therefore the values of the critical Mach number (stream Mach number for which $M_{max} = 1$) computed by formula (76) may be expected to be higher than the actual values.

CONCLUDING REMARKS

1. The method presented in this report reduces the problem of finding the velocity and pressure distribution along a preassigned symmetrical profile to the solution of a certain mapping problem which in turn is reduced to the solution of a functional equation by the method of successive approximations. Though the mathematical difficulties peculiar to the nonlinear differential equations of fluid dynamics are eliminated by this method, the practical application of this procedure requires the numerical solution of linear partial differential equations with variable coefficients. This last problem presents no theoretical difficulties, but really efficient methods for its solution are still in the process of development. Upon this development, which is intimately connected with the design of automatic computing machines, hinges the practical applicability of the present method.

2. The extension of the present method to the case of circulatory flows and to the case of flows with locally supersonic regions will require a more profound investigation of the "mappings conformal with respect to a compressible flow" introduced by the author in a previous report. (See reference 2.)

3. The author takes the opportunity to draw attention to the similarity between the method given in reference 2 and the one given (from a different point of view) in a paper by S. A. Christianovitch (reference 17). In the author's opinion, however, the paper by the noted Russian fluid dynamicist contains an error. In the terminology introduced in reference 2, a statement made by Christianovitch could be formulated as follows: "To every subsonic compressible flow past a profile P there exists a conjugate (modulo 1) incompressible flow past another profile P'." The author believes that this statement is misleading and that, for this reason, Christianovitch's method must fail in the case of circulatory flows.

Brown University,
Providence, R. I., September 1945.

APPENDIX I

CONCERNING THE COMPUTATION OF THE FUNCTION Φ

A. No matter which method is used for integrating the linear partial differential equation (50) satisfied by Φ it is convenient first to introduce the auxiliary complex variable $Z = X + iY$ by the relation

$$Z = \zeta + \frac{1}{\zeta} \tag{A1}$$

This transformation takes the domain $|\zeta| > 1$ into the domain exterior to the straight segment $(-1, +1)$. The functions Φ and T may be considered as functions of X and Y, and in view of the conformality of the transformation (A1) Φ satisfies the equation

$$\frac{\partial}{\partial X}\left(\frac{1}{T}\frac{\partial\Phi}{\partial X}\right) + \frac{\partial\Phi}{\partial Y}\left(\frac{1}{T}\frac{\partial\Phi}{\partial Y}\right) = 0 \tag{A2}$$

Furthermore, by virtue of (57)

$$\Phi\ (X, -Y) = \Phi\ (X,Y) \tag{A3}$$

This implies that

$$\left.\frac{\partial\Phi}{\partial Y}\right|_{Y = 0} = 0 \tag{A4}$$

for $X < -1$ and $X > 1$. But the same relation holds for $-1 < X < 1$ by virtue of (54). At infinity equation (55) implies that

$$\frac{\partial\Phi}{\partial X} \longrightarrow 1 \qquad \frac{\partial\Phi}{\partial Y} \longrightarrow 0 \qquad \text{as} \quad Z \longrightarrow \infty \tag{A5}$$

The determination of Φ requires the integration of equation (A2) in the half-plane $Y > 0$ under the boundary conditions (A4) and (A5).

A numerical or mechanical integration can be performed in a finite domain only. Therefore the upper half-plane is replaced by a sufficiently large rectangle with the vertices $(-A, 0)$, $(-A, B)$, (A, B), $(A, 0)$ and the condition (A5) is replaced by the boundary conditions

$$\Phi\ (\pm A,\ Y) = \pm A, \qquad\qquad \Phi\ (X,\ B) = X \tag{A6}$$

(Another way of passing to a finite domain would consist of a conformal transformation, say of $|\zeta| > 1$ into $|\zeta| < 1$.)

B. The boundary value stated formerly can be solved by means of electrical models. One method is that of the electrolytic bath due to Taylor and Sharman (references 10 and 11), the other that of network analogies due to Kron (references 12 and 13).

C. In order to solve equation (A2) numerically it is first replaced by a difference equation. (The possibility of reducing (A2) to an integral equation will not be discussed here.) Let X_0, Y_0 be a point of the domain considered and set $X_1 = X_0 + \delta$, $Y_1 = Y_0$, $X_2 = X_0$, $Y_2 = Y_0 + \delta$, $X_3 = X_0 - \delta$, $Y_3 = Y_0$, $X_4 = X_0$, $Y_4 = Y_0 - \delta$, δ being a small positive number. Denote the value of a function Ω at (x_i, y_i) by Ω_i and replace the expressions

$$\left(\frac{\partial\Omega}{\partial x}\right)_0, \quad \left(\frac{\partial\Omega}{\partial y}\right)_0, \quad \left(\frac{\partial^2\Omega}{\partial x^2}\right)_0, \quad \left(\frac{\partial^2\Omega}{\partial y^2}\right)_0$$

by

$$\frac{\Omega_1 - \Omega_3}{2\delta}, \quad \frac{\Omega_2 - \Omega_4}{2\delta}, \quad \frac{\Omega_1 + \Omega_3 - 2\Omega_0}{\delta^2}, \quad \frac{\Omega_3 + \Omega_4 - 2\Omega_0}{\delta^2}$$

respectively. Equation (A2) can be rewritten in the form

$$4T_0(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_0) = (T_1 - T_3)(\varphi_1 - \varphi_3) + (T_2 - T_4)(\varphi_2 - \varphi_4)$$

or

$$\varphi_0 = \beta_1\varphi_1 + \beta_2\varphi_2 + \beta_3\varphi_3 \tag{A7}$$

where

$$\left.\begin{aligned} \beta_1 &= \frac{4T_0 - T_1 + T_3}{16\ T_0} & \beta_2 &= \frac{4T_0 - T_2 + T_4}{16\ T_0} \\ \beta_3 &= \frac{4T_0 + T_1 - T_3}{16\ T_0} & \beta_4 &= \frac{4T_0 + T_2 - T_4}{16\ T_0} \end{aligned}\right\} \tag{A8}$$

If the point 0 lies on the line Y = 0, then the point 4 lies outside of the domain considered and according to (A3) the preceding formulas must be changed to

$$\left.\begin{aligned} \beta_1 &= \frac{4T_0 - T_1 + T_3}{16\ T_0} & \beta_2 &= \frac{1}{2} \\ \beta_3 &= \frac{4T_0 + T_1 - T_3}{16\ T} & \beta_4 &= 0 \end{aligned}\right\} \tag{A9}$$

Note that the following relation always holds

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$$

and that

$$\beta_i \geq 0 \tag{A10}$$

if δ is sufficiently small.

Since equation (A2) is invariant with respect to rotation, the same difference equation would be obtained if the square formed by the points 1, 2, 3, and 4 would not have its sides parallel to the coordinate axes. Therefore, it is easy to compute Φ at lattice points which do not form a uniform net. In fact, it is advisable to use more points near and on the segment (-1, 1) for here the function T will change rapidly and the values of Φ on this segment are the ones actually needed.

A simple computation shows that if the function

$$\Phi(X, Y) = \sum c_{nm} X^n Y^m$$

satisfies the differential equation (A2), then the polynomial

$$c_{00} + c_{10}X + c_{01}Y + c_{20}X^2 + c_{11}XY + c_{02}Y^2$$

satisfies the difference equation (A7). It also is known that solutions of the difference equation converge toward the solution of the differential equation as the parameter δ approaches zero.

D. The boundary value problem for the difference equation derived above can be solved either by a rigid iteration scheme, say by Liebmann's method (references 14 and 15) or by a relaxation method. Concerning the latter, reference is made to a recent paper by Emmons (reference 8) and to the literature quoted therein. The iteration methods require more computational labor than the consequent application of the relaxation method and the use of modern computing devices becomes imperative. On the other hand, the instructions for computing become simpler and the whole problem can be "coded" once and for all on an automatic computing machine. An attempt was made to solve equation (A2) by Liebmann's method using the I.B.M. Automatic Sequence Controlled Calculator. This attempt could not be considered successful because of the slowness of convergence. It is possible that convergence could be improved by using either a difference equation involving higher differences or a different iteration procedure. However, the design of automatic computing machines is now in the process of rapid development, and the speed at which these machines perform will doubtless be increased at a tremendous rate. This trend will, on the one hand, reduce the importance of the rate of convergence and, on the other, put a premium on the simplicity of the computational procedure.

E. In this section, the convergence of Liebmann's iteration method when applied to the boundary value problem for the difference equations (A7) to (A3) will be proved. The proof is patterned after the one given by Loewner for the case of the first boundary value problem for Laplace's difference equation. (See reference 15.)

Let $P_1, P_2, \ldots, P_N$ be the points at which the values of Φ (denoted by $\Phi_1, \Phi_2, \ldots, \Phi_N$) are to be computed. It will be assumed that among the four neighboring points of the point P_1 there is at least one boundary point (i.e., a point with either $x = \pm A$ or $Y = B$), and that one of the four neighboring points of P_n is $P_{n-1} (n = 2, 3, \ldots, N)$. Liebmann's procedure for finding the values Φ_i is as follows. Assume some trial values $\Phi_1^{(o)}, \Phi_2^{(o)}, \ldots \Phi_N^{(o)}$. Go over the points P_i in the indicated order correcting the value $\Phi_i^{(o)}$ using equations (A7) and (A8) (or equations (A7) and (A9) if P_i lies on the X-axis). In applying these formulas use corrected values of Φ whenever possible. In this way new values $\Phi_1^{(1)}, \Phi_2^{(1)}, \ldots, \Phi_N^{(1)}$ are obtained. Repeat this process to obtain the values $\Phi_i^{(2)}, \Phi_i^{(3)}, \ldots$. It is asserted that as $n \longrightarrow \infty$ $\Phi_i^{(n)}$ converges toward the desired value Φ_i.

In what follows it is assumed that the net is so fine that condition (A10) is satisfied. Plainly

$$\Phi_i^{(1)} = \sum_{i} \beta_{ij} \Phi_j^{(o)} + \beta_{io} \tag{A11}$$

and, in general,

$$\Phi_i^{(n+1)} = \sum_{j} \beta_{ij} \Phi_i^{(n)} + \beta_{io} \tag{A12}$$

where the constants β_{ij} are linear combinations of products of the β_i defined by (A8) and (A9) and the constants β_{io} depend upon the boundary conditions. Thus

$$\beta_{ij} \geq 0 \tag{A13}$$

Now assume that the boundary conditions read

$$\Phi = 0 \quad \text{for} \quad X = \pm A \quad \text{and for} \quad Y = B$$

and that $\Phi_i^{(o)} = 1$, $i = 1, 2, \ldots N$. It is easily seen that in this case all β_{io} vanish and that $\Phi_1^{(1)} < \Phi_1^{(o)}$, $\Phi_2^{(1)} < \Phi_2^{(o)}$, ... Since, in the case considered

$$\Phi_i^{(1)} = \sum_j \beta_{ij}$$

it follows that there exists a constant b such that

$$\sum_j \beta_{ij} < b < 1, \; i = 1, 2, \ldots, N \tag{A14}$$

Since, in the general case

$$\Phi_i^{(n)} = \sum \beta_{ij} \, \Phi_j^{(n-1)} + k_i \tag{A15}$$

then from (A12) and (A15) it follows that

$$\Phi_i^{(n+1)} - \Phi_i^{(n)} = \sum \beta_{ij} \left(\Phi^{(n)} - \Phi_j^{(n-1)}\right) \tag{A16}$$

Set

$$d_n = \max_i \left|\Phi_i^{(n)} - \Phi_i^{(n-1)}\right| \tag{A17}$$

From (A16) and (A13) it follows that

$$d_{n+1} \leq d_n \max_i \sum \beta_{ij}$$

so that by (A14)

$$d_{n+1} < b\, d_n$$

and hence

$$d_n < b^{n-1} d_1 \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty \tag{A18}$$

Thus the limits

$$\Phi_i = \lim_{n \to \infty} \Phi_i^{(n)} \tag{A19}$$

exist, since

$$\left| \Phi_i^{(n+m)} - \Phi_i^{(n)} \right| \leq \sum_{p=o}^{m-1} \left| \Phi_i^{(n+m-p)} - \Phi_i^{(n+m-p-1)} \right|$$

$$< \sum_{p=o}^{m-1} b^{n+m-p-1} d_1 = d_1 b^n \frac{1 - b^m}{1 - b} < \frac{d_1 b^n}{1 - b}$$

Plainly

$$\Phi_i = \sum \beta_{ij} \Phi_j + k_i \tag{A20}$$

which shows that the Φ_i satisfy both the boundary conditions and the difference equation.

Also, from (A12) and (A20) it follows that

$$\Phi_i^{(n+1)} - \Phi_i = \sum \beta_{ij} \left(\Phi_j^{(n)} - \Phi_j \right)$$

so that by (A13) and (A14)

$$\max_i \left| \Phi_i^{(n+1)} - \Phi_i \right| < \max_i \left| \Phi_i^{(n)} - \Phi_i \right| \qquad \text{(A21)}$$

This remark establishes the uniqueness of the solution.

Finally, if all differences $\Phi_i^{(n)} - \Phi_i^{(n-1)}$ are positive (negative), then by (A16) and (A13) so are all the differences $\Phi^{(n+1)} - \Phi_i^{(n)}$. This implies that all values Φ_i^n lie below (above) the desired values Φ_i.

To summarize, the following statements are seen to be true.

(i) Liebmann's method converges.

(ii) Both the maximum corrections $\left|\Phi_i^{(n+1)} - \Phi_i^{(n)}\right|$ and the maximum differences $\left|\Phi_i^{(n)} - \Phi_i\right|$ decrease monotonically.

(iii) If all corrections $\Phi_i^{(n+1)} - \Phi_i^{(n)}$ are positive (negative) all values $\Phi_i^{(n+1)}$ are smaller (larger) than the desired values.

If no assumptions concerning the order of the points P_i are made, a slight refinement of the argument shows that statements (i) and (iii) remain true and (ii) holds if the words "decrease monotonically" are replaced by "do not increase."

APPENDIX II

CONCERNING THE MAPPING FROM THE Z-PLANE INTO THE ζ-PLANE

A. This appendix contains the proof of two assertions concerning the transformation (62) made in section 7. First, the independence of the line integrals on the right-hand side of (62) on the path of integration will be established. Since Φ satisfies equation (50) there exists a function $\Psi(\xi,\eta)$ such that

$$\begin{aligned} \Phi_\xi &= T\,\Psi_\eta \\ \Phi_\eta &= -T\,\Psi_\xi \end{aligned} \tag{A22}$$

(The function Ψ is proportional to the stream-function of the compressible flow in the ζ-plane.) Since log w^* is an analytic function of ζ, Φ and Ψ may be considered as functions of log q^* and θ. Since equations (A22) are invariant with respect to conformal transformations, it follows that

$$\begin{aligned} \frac{\partial\Phi}{\partial\theta} &= T\,\frac{\partial\Psi}{\partial \log q^*} \\ \frac{\partial\Phi}{\partial \log q^*} &= -T\,\frac{\partial\Psi}{\partial\theta} \end{aligned} \tag{A23}$$

By introducing instead of log q^* the new variable q connected with log q^* by equation (3) and using equation (4), the following system is obtained

$$\begin{aligned} \frac{\partial\Phi}{\partial\theta} &= \frac{\rho_o q}{\rho}\,\frac{\partial\Psi}{\partial q} \\ \frac{\partial\Phi}{\partial q} &= -\,\frac{\rho_o(1 - M^2)}{\rho q}\,\frac{\partial\Psi}{\partial\theta} \end{aligned} \tag{A24}$$

(These are the well-known Chaplygin equations.)

Now (62) may be written in the form

$$x = C\int\frac{a_o}{q}\left\{\left(\cos\,\theta\Phi_\xi - \frac{\rho_o}{\rho}\sin\,\theta\Psi_\xi\right)d\xi + \left(\cos\,\theta\Phi_\eta - \frac{\rho_o}{\rho}\sin\,\theta\Psi_\eta\right)d\eta\right\}$$

$$y = C\int\frac{a_o}{q}\left\{\left(\sin\,\theta\Phi_\xi + \frac{\rho_o}{\rho}\cos\,\theta\Psi_\xi\right)d\xi + \left(\sin\,\theta\Phi_\eta + \frac{\rho_o}{\rho}\cos\,\theta\Psi_\eta\right)d\eta\right\}$$

or

$$\left.\begin{aligned} dx &= C\int \frac{a_o}{q}\left(\cos\theta\, d\Phi - \frac{\rho_o}{\rho}\sin\theta\, d\psi\right) \\ dy &= C\int \frac{a_o}{q}\left(\sin\theta\, d\Phi - \frac{\rho_o}{\rho}\cos\theta\, d\psi\right) \end{aligned}\right\} \quad \text{(A25)}$$

and equations (A24) are precisely the conditions that these integrals be independent of the path (cf. reference 16, p. 8).

B. It will now be shown that the mapping inverse to (62) is conformal with respect to the metric (34), (35). By use of the relations

$$\frac{\partial x}{\partial \xi} = J\frac{\partial \eta}{\partial y}, \quad \frac{\partial x}{\partial \eta} = -J\frac{\partial \xi}{\partial y}$$

$$\frac{\partial y}{\partial \xi} = -J\frac{\partial \eta}{\partial x}, \quad \frac{\partial y}{\partial \eta} = J\frac{\partial \xi}{\partial x}$$

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)}$$

together with (62) and (67), the following equations are obtained

$$\frac{\partial \xi}{\partial x} = (\Phi_\xi^2 + \Phi_\eta^2)(\sqrt{1 - M^2}\sin\theta\Phi_\eta + \cos\theta\Phi_\xi)$$

$$\frac{\partial \xi}{\partial y} = (\Phi_\xi^2 + \Phi_\eta^2)(-\sqrt{1 - M^2}\cos\theta\Phi_\eta + \sin\theta\Phi_\xi)$$

$$\frac{\partial \eta}{\partial x} = (\Phi_\xi^2 + \Phi_\eta^2)(-\sqrt{1 - M^2}\sin\theta\Phi_\xi + \cos\theta\Phi_\eta)$$

$$\frac{\partial \eta}{\partial y} = (\Phi_\xi^2 + \Phi_\eta^2)(\sqrt{1 - M^2}\cos\theta\Phi_\xi + \sin\theta\Phi_\eta)$$

It is seen that these partial derivatives satisfy the conditions

$$\frac{\partial \xi}{\partial x} = \frac{g_{11}\frac{\partial \eta}{\partial y} - g_{12}\frac{\partial \eta}{\partial x}}{\sqrt{g_{11}g_{22} - g_{12}^2}}$$

$$\frac{\partial \xi}{\partial y} = \frac{g_{12}\frac{\partial \eta}{\partial y} - g_{22}\frac{\partial \eta}{\partial x}}{\sqrt{g_{11}g_{22} - g_{12}^2}}$$

where the g_{ik} are given by (35). These are the well-known Beltrami equations expressing the conformality of the mapping

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

with respect to the metric (34).

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TABLE 1. q^{*2} and T as functions of $\sqrt{1-M^2}$

$\sqrt{1-M^2}$	$(q^*)^2$	T	$\sqrt{1-M^2}$	$(q^*)^2$	T
0.00	1.00000	0.00000	0.34	0.97656	0.51067
0.01	1.00000	0.01577	0.35	0.97432	0.52415
0.02	1.00000	0.03153	0.36	0.97194	0.53750
0.03	0.99999	0.04728	0.37	0.96939	0.55072
0.04	0.99996	0.06302	0.38	0.96670	0.56380
0.05	0.99993	0.07875	0.39	0.96383	0.57674
0.06	0.99988	0.09446	0.40	0.96079	0.58954
0.07	0.99981	0.11014	0.41	0.95758	0.60219
0.08	0.99971	0.12580	0.42	0.95417	0.61469
0.09	0.99959	0.14142	0.43	0.95057	0.62703
0.10	0.99944	0.15701	0.44	0.94676	0.63922
0.11	0.99926	0.17256	0.45	0.94275	0.65125
0.12	0.99903	0.18807	0.46	0.93852	0.66312
0.13	0.99877	0.20353	0.47	0.93406	0.67482
0.14	0.99846	0.21894	0.48	0.92936	0.68635
0.15	0.99810	0.23429	0.49	0.92443	0.69772
0.16	0.99769	0.24959	0.50	0.91923	0.70891
0.17	0.99722	0.26482	0.51	0.91378	0.71992
0.18	0.99670	0.27999	0.52	0.90806	0.73075
0.19	0.99611	0.29509	0.53	0.90205	0.74140
0.20	0.99545	0.31012	0.54	0.89575	0.75187
0.21	0.99472	0.32506	0.55	0.88915	0.76215
0.22	0.99391	0.33993	0.56	0.88223	0.77223
0.23	0.99302	0.35471	0.57	0.87499	0.78213
0.24	0.99204	0.36940	0.58	0.86741	0.79183
0.25	0.99098	0.38400	0.59	0.85948	0.80134
0.26	0.98982	0.39851	0.60	0.85119	0.81064
0.27	0.98856	0.41291	0.61	0.84252	0.81975
0.28	0.98720	0.42722	0.62	0.83346	0.82865
0.29	0.98573	0.44141	0.63	0.82401	0.83734
0.30	0.98415	0.45550	0.64	0.81413	0.84583
0.31	0.98244	0.46947	0.65	0.80382	0.85410
0.32	0.98061	0.48332	0.66	0.79306	0.86216
0.33	0.97865	0.49706	0.67	0.78184	0.87001

TABLE 1. (Continued)

$\sqrt{1-M^2}$	$(q^*)^2$	T
0.68	0.77014	0.87765
0.69	0.75794	0.88506
0.70	0.74523	0.89226
0.71	0.73198	0.89923
0.72	0.71817	0.90598
0.73	0.70380	0.91251
0.74	0.68882	0.91881
0.75	0.67324	0.92489
0.76	0.65701	0.93073
0.77	0.64012	0.93635
0.78	0.62255	0.94173
0.79	0.60427	0.94688
0.80	0.58525	0.95180
0.81	0.56546	0.95648
0.82	0.54490	0.96093
0.83	0.52351	0.96514
0.84	0.50127	0.96911
0.85	0.47814	0.97285
0.86	0.45411	0.97634
0.87	0.42913	0.97960
0.88	0.40316	0.98262
0.89	0.37618	0.98540
0.90	0.34813	0.98793
0.91	0.31900	0.99023
0.92	0.28872	0.99228
0.93	0.25726	0.99409
0.94	0.22458	0.99566
0.95	0.19062	0.99699
0.96	0.15535	0.99807
0.97	0.11870	0.99892
0.98	0.08063	0.99952
0.99	0.04108	0.99988
1.00	0.00000	1.00000

TABLE 2. $\sqrt{1-M^2}$ and T as functions of q^{*2}

$(q^*)^2$	$\sqrt{1-M^2}$	T	$(q^*)^2$	$\sqrt{1-M^2}$	T
0.000	1.00000	1.00000	0.170	0.95589	0.99766
0.005	0.99880	1.00000	0.175	0.95448	0.99750
0.010	0.99760	0.99999	0.180	0.95305	0.99734
0.015	0.99639	0.99998	0.185	0.95162	0.99718
0.020	0.99518	0.99997	0.190	0.95018	0.99701
0.025	0.99396	0.99996	0.195	0.94873	0.99683
0.030	0.99274	0.99994	0.200	0.94728	0.99665
0.035	0.99151	0.99991	0.205	0.94581	0.99646
0.040	0.99027	0.99989	0.210	0.94434	0.99626
0.045	0.98903	0.99986	0.215	0.94286	0.99606
0.050	0.98776	0.99982	0.220	0.94137	0.99586
0.055	0.98652	0.99978	0.225	0.93987	0.99564
0.060	0.98526	0.99974	0.230	0.93837	0.99542
0.065	0.98400	0.99969	0.235	0.93685	0.99519
0.070	0.98273	0.99964	0.240	0.93533	0.99496
0.075	0.98145	0.99959	0.245	0.93380	0.99471
0.080	0.98016	0.99953	0.250	0.93226	0.99446
0.085	0.97887	0.99946	0.255	0.93070	0.99421
0.090	0.97757	0.99939	0.260	0.92914	0.99394
0.095	0.97627	0.99932	0.265	0.92758	0.99367
0.100	0.97496	0.99924	0.270	0.92600	0.99339
0.105	0.97364	0.99916	0.275	0.92441	0.99311
0.110	0.97232	0.99908	0.280	0.92281	0.99281
0.115	0.97099	0.99899	0.285	0.92120	0.99251
0.120	0.96965	0.99889	0.290	0.91958	0.99220
0.125	0.96831	0.99879	0.295	0.91796	0.99188
0.130	0.96696	0.99868	0.300	0.91632	0.99155
0.135	0.96560	0.99857	0.305	0.91467	0.99121
0.140	0.96424	0.99846	0.310	0.91301	0.99087
0.145	0.96286	0.99834	0.315	0.91134	0.99051
0.150	0.96148	0.99821	0.320	0.90966	0.99015
0.155	0.96010	0.99808	0.325	0.90797	0.98978
0.160	0.95870	0.99794	0.330	0.90627	0.98940
0.165	0.95730	0.99780	0.335	0.90455	0.98900
			0.340	0.90283	0.98860

TABLE 2. (Continued)

$(q*)^2$	$\sqrt{1-M^2}$	T	$(q*)^2$	$\sqrt{1-M^2}$	T
0.345	0.90109	0.98819	0.500	0.84056	0.96933
0.350	0.89935	0.98777	0.505	0.83835	0.96847
0.355	0.89759	0.98734	0.510	0.83612	0.96760
0.360	0.89582	0.98690	0.515	0.83387	0.96671
0.365	0.89403	0.98645	0.520	0.83160	0.96579
0.370	0.89224	0.98598	0.525	0.82931	0.96486
0.375	0.89043	0.98551	0.530	0.82701	0.96390
0.380	0.88861	0.98502	0.535	0.82468	0.96293
0.385	0.88677	0.98453	0.540	0.82232	0.96193
0.390	0.88492	0.98402	0.545	0.81995	0.96091
0.395	0.88306	0.98350	0.550	0.81756	0.95986
0.400	0.88119	0.98296	0.555	0.81514	0.95880
0.405	0.87930	0.98242	0.560	0.81270	0.95770
0.410	0.87740	0.98186	0.565	0.81023	0.95659
0.415	0.87549	0.98129	0.570	0.80774	0.95545
0.420	0.87356	0.98070	0.575	0.80523	0.95428
0.425	0.87161	0.98011	0.580	0.80269	0.95308
0.430	0.86966	0.97949	0.585	0.80013	0.95186
0.435	0.86768	0.97887	0.590	0.79754	0.95061
0.440	0.86569	0.97823	0.595	0.79492	0.94933
0.445	0.86369	0.97757	0.600	0.79228	0.94802
0.450	0.86167	0.97691	0.605	0.78961	0.94668
0.455	0.85964	0.97622	0.610	0.78691	0.94531
0.460	0.85758	0.97552	0.615	0.78418	0.94391
0.465	0.85552	0.97481	0.620	0.78142	0.94247
0.470	0.85343	0.97408	0.625	0.77863	0.94101
0.475	0.85133	0.97333	0.630	0.77581	0.93950
0.480	0.84921	0.97256	0.635	0.77296	0.93796
0.485	0.84708	0.97178	0.640	0.77007	0.93639
0.490	0.84492	0.97098	0.645	0.76715	0.93477
0.495	0.84275	0.97016	0.650	0.76420	0.93312

TABLE 2. (Continued)

$(q^*)^2$	$\sqrt{1-M^2}$	T
0.655	0.76121	0.93142
0.660	0.75819	0.92969
0.665	0.75513	0.92791
0.670	0.75203	0.92609
0.675	0.74889	0.92422
0.680	0.74571	0.92231
0.685	0.74249	0.92035
0.690	0.73923	0.91833
0.695	0.73592	0.91627
0.700	0.73257	0.91415
0.705	0.72918	0.91198
0.710	0.72573	0.90975
0.715	0.72224	0.90747
0.720	0.71870	0.90512
0.725	0.71511	0.90271
0.730	0.71146	0.90023
0.735	0.70775	0.89768
0.740	0.70399	0.89507
0.745	0.70017	0.89238
0.750	0.69629	0.88962
0.755	0.69235	0.88677
0.760	0.68834	0.88385
0.765	0.68427	0.88084
0.770	0.68012	0.87774
0.775	0.67590	0.87454
0.780	0.67160	0.87125
0.785	0.66723	0.86786
0.790	0.66277	0.86436
0.795	0.65823	0.86075
0.800	0.65360	0.85703
0.805	0.64888	0.85318
0.810	0.64406	0.84921
0.815	0.63914	0.84510
0.820	0.63411	0.84085
0.825	0.62897	0.83645
0.830	0.62371	0.83190
0.835	0.61833	0.82718
0.840	0.61283	0.82228
0.845	0.60718	0.81720
0.850	0.60140	0.81193
0.855	0.59546	0.80644
0.860	0.58936	0.80073
0.865	0.58309	0.79479
0.870	0.57663	0.78859
0.875	0.56998	0.78212
0.880	0.56313	0.77535
0.885	0.55605	0.76827
0.890	0.54874	0.76086
0.895	0.54116	0.75307
0.900	0.53331	0.74488
0.905	0.52515	0.73626
0.910	0.51666	0.72715
0.915	0.50781	0.71752
0.920	0.49856	0.70730
0.925	0.48886	0.69643
0.930	0.47867	0.68483
0.935	0.46793	0.67241
0.940	0.45655	0.65905
0.945	0.44446	0.64460
0.950	0.43152	0.62890
0.955	0.41761	0.61171
0.960	0.40252	0.59274
0.965	0.38600	0.57159
0.970	0.36769	0.54767
0.975	0.34704	0.52017
0.980	0.32320	0.48774
0.985	0.29471	0.44806
0.990	0.25852	0.39636
0.995	0.20623	0.31944
1.000	0.00000	0.00000

TABLE 3. $M_1 \rho/\rho_0$, p/p_0, and q/a_0 as functions of q^{*2}

$(q^*)^2$	M	ρ/ρ_0	p/p_0	q/a_0
0.00	0.0000	1.0000	1.0000	0.0000
0.02	0.0982	0.9952	0.9933	0.0981
0.04	0.1392	0.9904	0.9865	0.1389
0.06	0.1711	0.9855	0.9797	0.1706
0.08	0.1982	0.9806	0.9729	0.1974
0.10	0.2224	0.9757	0.9660	0.2213
0.12	0.2445	0.9707	0.9591	0.2430
0.14	0.2650	0.9657	0.9522	0.2631
0.16	0.2844	0.9607	0.9452	0.2821
0.18	0.3028	0.9556	0.9382	0.3000
0.20	0.3204	0.9505	0.9311	0.3171
0.22	0.3374	0.9453	0.9240	0.3336
0.24	0.3538	0.9401	0.9168	0.3494
0.26	0.3678	0.9348	0.9096	0.3648
0.28	0.3852	0.9295	0.9024	0.3795
0.30	0.4005	0.9241	0.8951	0.3942
0.32	0.4154	0.9187	0.8878	0.4083
0.34	0.4300	0.9132	0.8803	0.4222
0.36	0.4444	0.9077	0.8728	0.4358
0.38	0.4587	0.9021	0.8653	0.4492
0.40	0.4728	0.8965	0.8576	0.4625
0.42	0.4867	0.8908	0.8500	0.4754
0.44	0.5005	0.8850	0.8422	0.4883
0.46	0.5143	0.8791	0.8344	0.5011
0.48	0.5281	0.8732	0.8265	0.5138
0.50	0.5417	0.8672	0.8185	0.5263
0.52	0.5554	0.8611	0.8104	0.5388
0.54	0.5690	0.8549	0.8022	0.5512
0.56	0.5827	0.8486	0.7940	0.5637
0.58	0.5964	0.8422	0.7856	0.5760
0.60	0.6102	0.8357	0.7771	0.5884
0.62	0.6240	0.8291	0.7685	0.6008
0.64	0.6380	0.8224	0.7598	0.6132
0.66	0.6520	0.8155	0.7509	0.6256
0.68	0.6663	0.8085	0.7418	0.6382

TABLE 3. (Continued)

$(q^*)^2$	M	ρ/ρ_o	p/p_o	q/a_o
0.70	0.6807	0.8014	0.7326	0.6508
0.72	0.6953	0.7940	0.7232	0.6636
0.74	0.7102	0.7865	0.7136	0.6765
0.76	0.7254	0.7788	0.7038	0.6896
0.78	0.7409	0.7709	0.6937	0.7029
0.80	0.7568	0.7626	0.6834	0.7164
0.82	0.7733	0.7541	0.6727	0.7303
0.84	0.7903	0.7454	0.6617	0.7447
0.86	0.8079	0.7360	0.6501	0.7600
0.88	0.8264	0.7263	0.6381	0.7746
0.90	0.8459	0.7160	0.6253	0.7906
0.92	0.8669	0.7049	0.6118	0.8076
0.94	0.8897	0.6927	0.5970	0.8260
0.96	0.9154	0.6791	0.5806	0.8464
0.98	0.9463	0.6627	0.5609	0.8706
1.00	1.0000	0.6342	0.5274	0.9119

TABLE 4. q^{*2} as a function of M

M	$(q^*)^2$	M	$(q^*)^2$
0.00	0.0000	0.50	0.4391
0.02	0.0008	0.52	0.4682
0.04	0.0033	0.54	0.4975
0.06	0.0075	0.56	0.5268
0.08	0.0140	0.58	0.5561
0.10	0.0207	0.60	0.5853
0.12	0.0298	0.62	0.6142
0.14	0.0404	0.64	0.6439
0.16	0.0526	0.66	0.6712
0.18	0.0663	0.68	0.6991
0.20	0.0814	0.70	0.7262
0.22	0.0979	0.72	0.7528
0.24	0.1158	0.74	0.7787
0.26	0.1350	0.76	0.8039
0.28	0.1553	0.78	0.8280
0.30	0.1776	0.80	0.8512
0.32	0.1995	0.82	0.8732
0.34	0.2233	0.84	0.8941
0.36	0.2477	0.86	0.9136
0.38	0.2732	0.88	0.9317
0.40	0.2994	0.90	0.9483
0.42	0.3263	0.92	0.9632
0.44	0.3538	0.94	0.9763
0.46	0.3819	0.96	0.9872
0.48	0.4103	0.98	0.9954
		1.00	1.0000

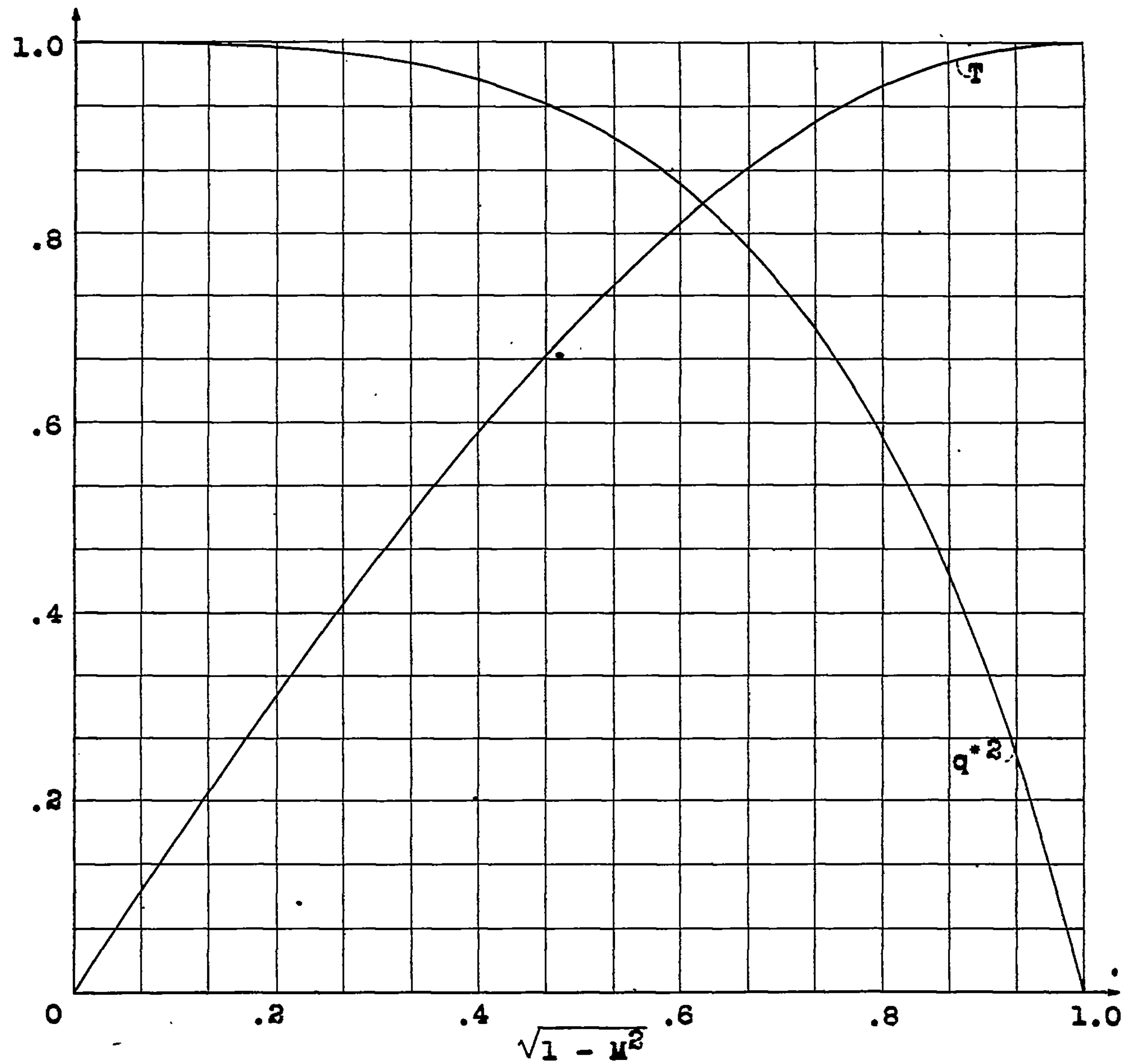

Figure 1

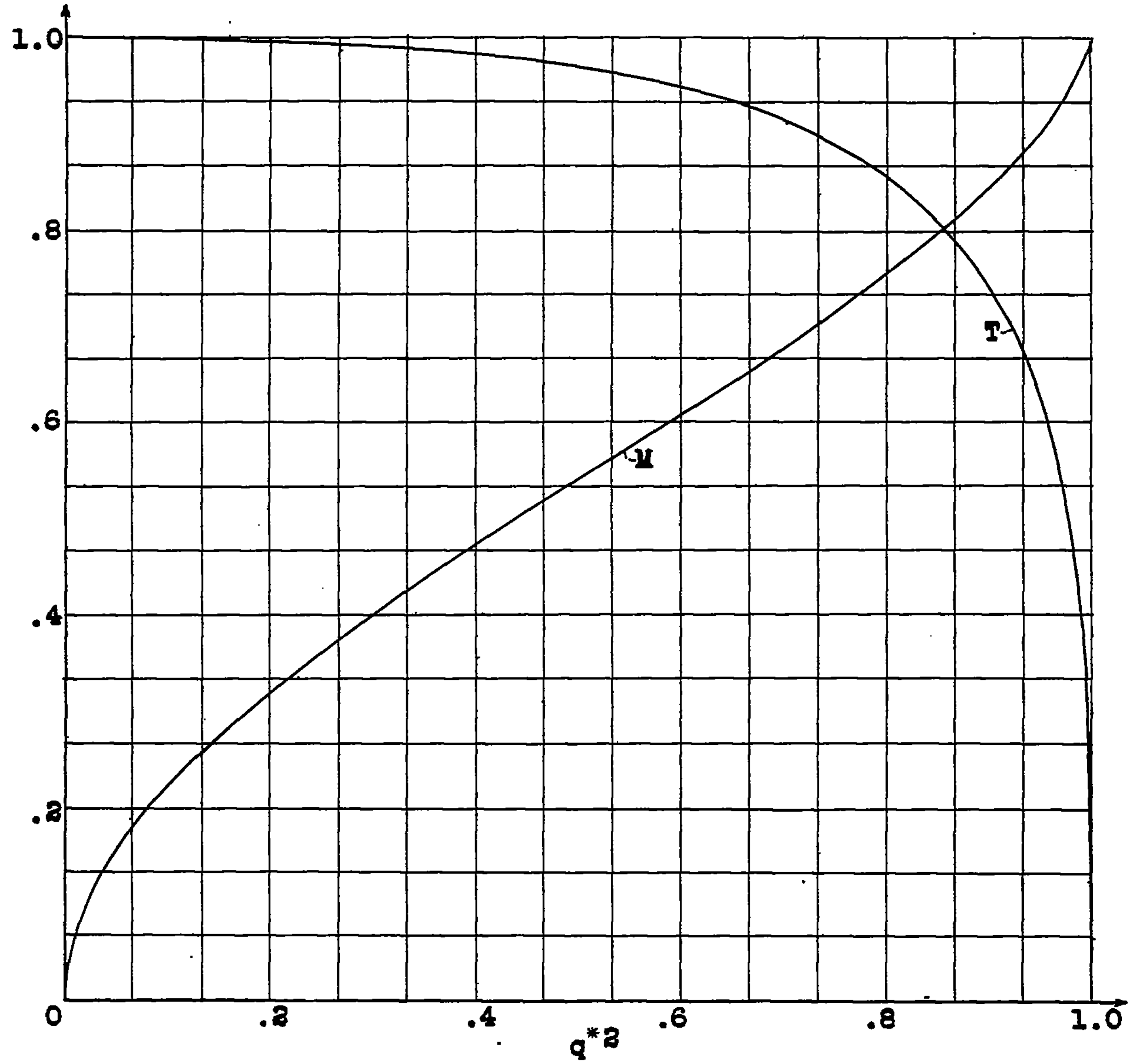

Figure 2

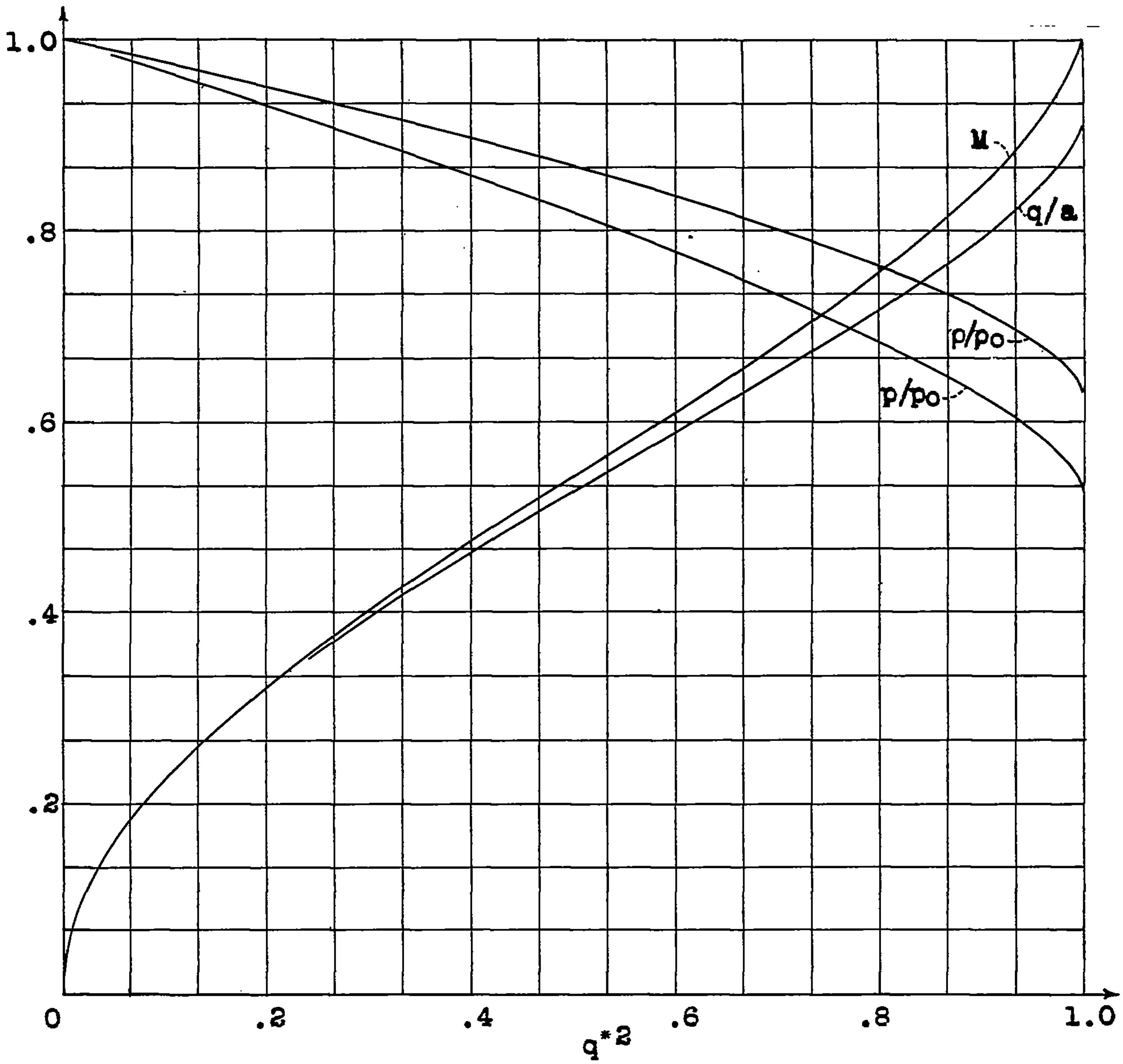

Figure 3

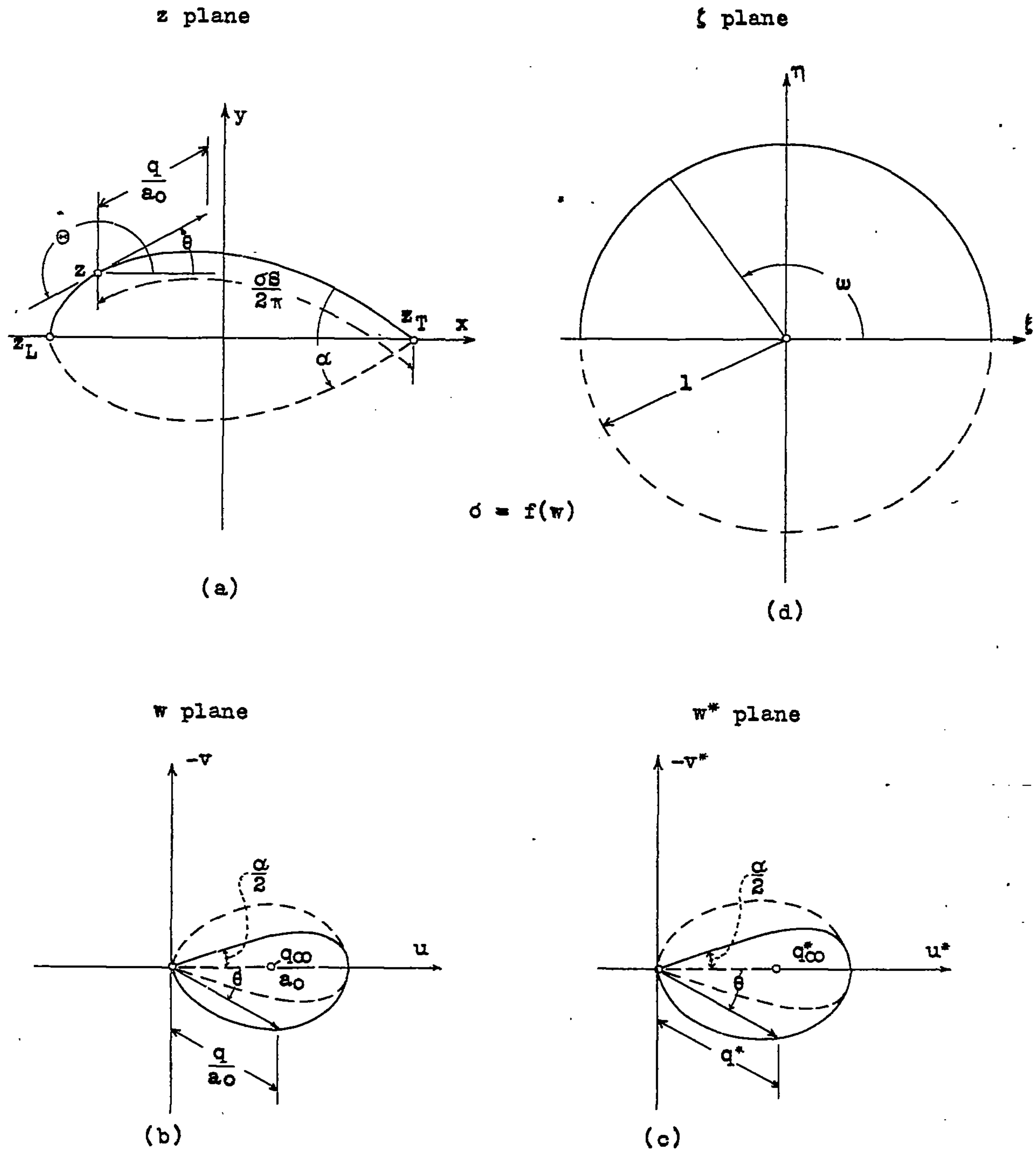
z plane
ζ plane
y
q/a₀
Θ
θ
z
σS/2π
z_T
x
z_L
α
(a)
η
ω
ξ
1
σ = f(w)
(d)
w plane
-v
α/2
u
q∞/a₀
θ
q/a₀
(b)
w* plane
-v*
α/2
u*
q*∞
θ
q*
(c)

Figure 4